ISRAEL JOURNAL OF MATHEMATICS, Vol. 69, No. 1, 1990

SKEW PRODUCTS OVER THE IRRATIONAL ROTATION

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ABSTRACT

Here we give conditions on a class of functions defining skew product extensions of irrational rotations on **T** which ensure ergodicity. These results produce extensions of the work done by P. Hellekalek and G. Larcher [HL1] and [HL2] to the larger class of functions which are piecewise absolutely continuous, have zero integral and have a derivative which is Riemann integrable with a non-zero integral.

1. Introduction

In this paper we study skew product extensions of irrational rotations on **T**, where the extension is defined by the class of real-valued functions which are piecewise absolutely continuous, have zero integral and have a derivative which is Riemann integrable with non-zero integral. We study this class of functions and show that their properties ensure that the skew product is ergodic.

This paper follows on from the work of P. Hellekalek and G. Larcher [HL1]. In [HL2] certain conditions were assumed on the irrational number used and, as also in [HL1], the function was permitted to have only one discontinuity. In [HL2] Hellekalek and Larcher approach the subject from the point of view of Fourier coefficients which yields the condition on the irrational number.

Some results in this area have also been obtained by Oren [O], who demonstrated ergodicity for a class of step functions.

In Section 2, firstly we discuss the orbits of the irrational rotation and give

† Supported by SERC Grant No. 85318881.
Received March 1, 1989

some definitions and notation; then we give a sequence of lemmas about functions in our class. We demonstrate that the only possibility for their group of essential values (or asymptotic range) is **R**, which guarantees ergodicity of the skew product (see [S]).

2. Conditions for ergodicity

We shall use T to denote irrational rotation on $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ by $\alpha \notin \mathbf{Q}$ whose continued fraction expansion partial quotient denominators are $\{q_k\}$ for $k \geqq 0$. μ will denote Lebesgue measure on **T**, whose Borel field will be denoted by Ω. For small sub-intervals of **T** we shall consider the order inherited from **R**, and use the words left and right accordingly.

For $x \in \mathbf{T}$ we denote by $\| x \|$ the distance of x from the nearest integer. The following result may be found in section 4 of [K].

2.1. Proposition. *Let $P_n(\alpha)$ be the set of right half-open partition intervals of* **T** *defined by the points $\{-j \cdot \alpha\}$ for $j = 0, \ldots, q_n - 1$. Then each interval of $P_n(\alpha)$ has length $\| q_{n-1}\alpha \|$ or $\| q_n\alpha \| + \| q_{n-1}\alpha \|$.*

The following shorthand will be used frequently throughout this paper:

$$f_n(x) = \sum_{k=0}^{n-1} f \circ T^k(x)$$

where f is any Borel function on **T**, and $n \geqq 1$. Note that if we also define $f_0 \equiv 0$ and

$$f_n = -f_{-n}(T^n x)$$

for $n \leqq -1$, then this defines a cocycle for T.

2.2. Convention. Throughout this section we assume that $f: \mathbf{T} \to \mathbf{R}$ is piecewise absolutely continuous, that is, there are finitely many points $\omega_i \in \mathbf{T}$ for $i = 1, \ldots, N$, with $0 \leqq \omega_1 < \cdots < \omega_N < 1$ such that f is absolutely continuous on each interval $[\omega_i, \omega_{i+1})$ for $i = 1, \ldots, N$. We interpret $[\omega_N, \omega_{N+1})$ as $[\omega_N, \omega_1)$.

We may immediately assume that $\omega_1 = 0$ since the rotation of the domain of f necessary to bring this about may be performed initially, and does not affect the ergodicity in question. We also assume that $\int f d\mu = 0$, that f' is Riemann integrable and that $\int f' d\mu > 0$.

2.3. LEMMA. *There are positive constants K_1, K_2 and a positive integer Z such that for $n > Z$ we have*

$$n \cdot K_1 < f'_n(x) < n \cdot K_2$$

for $\mu - a.e.\ x \in \mathbf{T}$.

PROOF. From the definition of f and using Weyl's Theorem we have that

$$\frac{1}{n} \sum_{i=0}^{n-1} f' \circ T^i(x) \to \int f' d\mu$$

uniformly, for all $x \in \mathrm{T}$. Given this, there is a positive integer Z, such that for $n > Z$ we obtain

$$\left| \frac{f'_n(x)}{n} - \int f' \cdot d\mu \right| < \frac{1}{2} \int f' d\mu$$

for all $x \in \mathrm{T}$. So

$$K_1 < \frac{f'_n(x)}{n} < K_2$$

for all $x \in \mathrm{T}$, *where* $K_1 = \frac{1}{2} \int f' d\mu$ and $K_2 = \frac{3}{2} \int f' d\mu$. □

Suppose now that at $0 = \omega_1 < \cdots < \omega_N < 1$, f jumps by $d_i = f(\omega_i^+) - f(\omega_i^-)$ for $i = 1, \ldots, N$, where $f(\omega_i^+)$, $f(\omega_i^-)$ are the limits of $f(x)$ as x approaches ω_i from the left, right respectively. Then, clearly since f is piecewise absolutely continuous we have the following relationship between $\int f' d\mu$ and these jumps:

2.4. LEMMA. $\int f' d\mu = \Sigma_{i=1}^N d_i$.

Hence the condition for $\int f' d\mu \neq 0$ is equivalent to $\Sigma\, d_i \neq 0$. We also have the following:

2.5. PROPOSITION. *The graph of f_{q_n} has discontinuities at $\{\omega_i - j \cdot \alpha\}$ for $0 \leqq j < q_n$, with jumps of size d_i at these points, for $i = 1, \ldots, N$.*

We now give results about the behaviour of f_{q_n} within the intervals of the partition $P_n(\alpha)$ defined in 2.1.

2.6. LEMMA. *For all n, and each interval Q_i^n of $P_n(\alpha)$ for $i = 1, \ldots, q_n - 1$, there is a subinterval $J_i^n \subseteq Q_i^n$ on which f_{q_n} is continuous, and which satisfies*

$$\mu(J_i^n) \geqq \frac{1}{2N-2} \cdot \mu(Q_i^n).$$

PROOF. Firstly we show that for each discontinuity ω_i for $i = 2, \ldots, N$ of f there can be at most two discontinuities of f_{q_n} due to it in Q_i^n.

By 2.5, the discontinuities of f_{q_n} due to ω_i occur at the points $\{\omega_i - j \cdot \alpha\}$ for $0 \leqq j < q_n$. Their spacing is the same as for $P_n(\alpha)$ (since this is the set $P_n(\alpha)$ shifted by ω_i). The distance between two discontinuities due to ω_i is thus $\| q_{n-1}\alpha \|$ or $\| q_n\alpha \| + \| q_{n-1}\alpha \|$ from 2.1. These distances are also the two possible sizes of Q_i^n. Now we have that

$$\| q_{n-1}\alpha \| < \| q_{n-1}\alpha \| + \| q_n\alpha \| < 2 \| q_{n-1}\alpha \|.$$

We see that if Q_i^n has length $\| q_{n-1}\alpha \| + \| q_n\alpha \|$, then at most two discontinuities due to ω_i of separation $\| q_{n-1}\alpha \|$ may be placed inside the interval. If Q_i^n has length $\| q_{n-1}\alpha \|$, then clearly only at most one discontinuity due to ω_i may fall inside it. Thus the maximum number of discontinuities of f_{q_n} which may occur in any interval of $P_n(\alpha)$ is $2(N-1)$. This implies the assertion. □

2.7. LEMMA. *There exist strictly positive constants H, F and an integer N_1 with the following properties: Within each interval Q_i^n of $P_n(\alpha)$ for $i = 1, \ldots, q_n - 1$, there is a subinterval $J_i^n = [a_i^n, b_i^n)$ such that for $n > N_1$*

(i) *f_{q_n} rises through a height greater than H on J_i^n,*

(ii) *for any interval $I_i^n = [x, y]$ in $f_{q_n}(J_i^n)$ we have that*

$$\frac{\mu(f_{q_n}^{-1}(I_i^n) \cap J_i^n)}{\mu(Q_i^n)} > F|y - x|.$$

PROOF. For any interval Q_i^n of $P_n(\alpha)$, by 2.6 we may consider a subinterval $J_i^n = [a_i^n, b_i^n) \subseteq Q_i^n$ on which f_{q_n} is continuous, and which satisfies

$$\mu(J_i^n) \geqq \frac{1}{2N-2} \cdot \mu(Q_i^n).$$

By rearranging and using 2.1 we see that

$$|b_i^n - a_i^n| \geqq \frac{1}{2N-2}\,\mu(Q_i^n) \geqq \frac{1}{2N-2}\,\| q_{n-1}\alpha \| \tag{2.1}$$

for all n. From 2.3, for $n > N_1$ and all $x \in \mathbb{T}$ we have that

(2.2) $$q_nK_2 > f'_{q_n}(x) > q_nK_1.$$

Now we choose

$$H = \frac{1}{4}\,\frac{K_1}{2N-2}.$$

By (2.2) f_{q_n} is strictly increasing on J_i^n for $n > N_1$ and

$$h = f_{q_n}(b_i^n) - f_{q_n}(a_i^n) = \int_{a_i^n}^{b_i^n} f'_{q_n} d\mu.$$

In order to obtain a lower bound for this height we use (2.1) and (2.2) to give us

$$h > \frac{K_1}{2N-2}\, q_n \,\| q_{n-1}\alpha \|,$$

from which we get

$$h > \frac{1}{2}\,\frac{K_1}{2N-2}$$

for $n > N_1$. This proves (i).

For (ii), define $m = f_{q_n}^{-1}(x) \cap J_i^n$ and $p = f_{q_n}^{-1}(y) \cap J_i^n$; then, since f_{q_n} is strictly increasing for $n > N_1$, we have that $f_{q_n}^{-1}(I_i^n) \cap J_i^n = [m, p)$, and

$$y - x = \int_m^p f'_{q_n} d\mu.$$

From (2.2) we get, for $n > N_1$, that

$$p - m > \frac{y-x}{K_2 \cdot q_n}.$$

Now from 2.1 we have that

$$\mu(Q_i^n) \leqq \| q_{n-1}\alpha \| + \| q_n\alpha \| < 2 \,\| q_{n-1}\alpha \|.$$

Hence

$$\frac{p-m}{\mu(Q_i^n)} > \frac{y-x}{2K_2 \cdot q_n \cdot \| q_{n-1}\alpha \|}$$

and for $n > N_1$,

$$\frac{\mu(f_{q_n}^{-1}(I_i^n) \cap J_i^n)}{\mu(Q_i^n)} > \frac{y-x}{2K_2}.$$

Putting $F = 1/2K_2$ we note that F is strictly positive and independent of i, so this completes the proof of (ii). □

2.8. LEMMA. *For any $A \subseteq \mathbf{T}$ with $\mu(A) > 0$, and every $\varepsilon > 0$, there is an $A_0 \subseteq A$ with $\mu(A \setminus A_0) < \varepsilon$ and an infinite sequence $\{q_{k_i}\}$ of partial quotient denominators such that $T^{q_{k_i}} x \in A$ for all i, and all $x \in A_0$.*

PROOF. For any Borel set B of positive measure the map

$$x \mapsto \mu(B \cap (B - x))$$

is continuous at 0, where $B - x$ denotes the set B translated by the element $-x \in \mathbf{T}$ (c.f. [R, Theorem 1.1.5]).

So, given $\varepsilon > 0$, there is a $\delta > 0$ such that $|\mu(B) - \mu(B \cap (B - x))| < \varepsilon$ for $|x| < \delta$. Now, $\| q_k \alpha \| \to 0$ as $k \to \infty$, so given $\delta > 0$ we may choose $K > 0$ such that, for $k > K$, we have $\| q_k \alpha \| < \delta$, and hence

$$|\mu(B) - \mu(B \cap (B - x))| = |\mu(B) - \mu(B \cap T^{-q_k} B)| < \varepsilon.$$

If we apply the above to our set A, given $\varepsilon > 0$ we can find an n_1 such that

$$\mu(A \cap T^{-q_{n_1}} A) > \mu(A) - \varepsilon/2. \tag{2.3}$$

Now we apply the above argument to the set $A \cap T^{-q_{n_1}} A$ and obtain an n_2 with

$$\mu(A \cap T^{-q_{n_1}} A \cap T^{-q_{n_2}} A \cap T^{-q_{n_1} - q_{n_2}} A) > \mu(A \cap T^{-q_{n_1}} A) - \varepsilon/4. \tag{2.4}$$

Then by monotonicity, (2.3) and (2.4) we have that

$$\begin{aligned} \mu(A \cap T^{-q_{n_1}} A \cap T^{-q_{n_2}} A) &\geqq \mu(A \cap T^{-q_{n_1}} A \cap T^{-q_{n_2}} A \cap T^{-q_{n_1} - q_{n_2}} A) \\ &> \mu(A \cap T^{-q_{n_1}} A) - \varepsilon/4 > \mu(A) - 3\varepsilon/4. \end{aligned}$$

Inductively, we get for $i \geqq 1$

$$\mu(A \cap T^{-q_{n_1}} A \cap \cdots \cap T^{-q_{n_i}} A) > \mu(A) - \frac{(2^i - 1)\varepsilon}{2^i}.$$

Letting $i \to \infty$ and defining $q_{n_0} = 0$, we have that

$$\mu\left(\bigcap_{i=0}^{\infty} T^{-q_{n_i}} A \right) > \mu(A) - \varepsilon.$$

The set $A_0 = \bigcap_{i=0}^{\infty} T^{-q_{n_i}} A$ has the required properties. □

2.9. LEMMA. *Given $A \subseteq \mathbf{T}$ with $\mu(A) > 0$, and $\varepsilon' > 0$, there is an $A_0 \subseteq A$ with $\mu(A_0) > 0$ and $\mu(A \setminus A_0) < \varepsilon'$, with the following properties: Given $\varepsilon > 0$,* *there exists a* $\delta > 0$ *such that, for all* $x \in A_0$, *and all intervals* $I(x)$ *containing* x *of length* $\mu(I(x)) < \delta$, *we have that*

$$\frac{\mu(I(x) \cap A)}{\mu(I(x))} > 1 - \varepsilon.$$

PROOF. Define, for $n \geqq 1$, a sequence of functions g_n on $\mathbf{T}$ by

$$g_n(x) = \inf \frac{\mu(I_n(x) \cap A)}{\mu(I_n(x)}$$

where the infimum is taken over all intervals $I_n(x) \subset \mathbf{T}$ containing x which satisfy $\mu(I_n(x)) < 1/n$. By a similar argument to that given in the proof of 2.8, the function

$$g(x, \beta, \gamma) = \frac{\mu((x - \beta, x + \gamma) \cap A)}{\gamma + \beta}$$

is continuous in x, for any $\beta, \gamma > 0$. Now g_n defined above is the infimum of such functions, with the condition that $|\gamma + \beta| < 1/n$. From [C, p. 229] we see that g_n is upper semi-continuous, and therefore Borel measurable.

The Lebesgue Density Theorem says that $g_n(x) \to 1$ for almost all $x \in A$ and $g_n(x) \to 0$ for almost all $x \in A^c$. Hence the Borel function $g = \liminf g_n$ is almost everywhere equal to the characteristic function of A.

By Egoroff's Theorem, given $\varepsilon' > 0$, there is an $A_0 \subseteq A$ with $\mu(A \setminus A_0) < \varepsilon'$ such that $g_n \to g$ uniformly on A_0. That is, given $\varepsilon > 0$, there is an $N \in \mathbf{N}$ such that

$$|g_n(x) - g(x)| < \varepsilon$$

for all $x \in A_0$, and all $n \geqq N$. Thus we have for such n

$$g_n(x) > 1 - \varepsilon$$

for all $x \in A_0$, since g is the characteristic function of A. By definition of g_n this implies that for $n \geqq N$

$$\inf \frac{\mu(I_n(x)) \cap A)}{\mu(I_n(x))} > 1 - \varepsilon.$$

So, putting $\delta = 1/N$, we have our result. □

2.10. LEMMA. *$E(f) \neq \lambda\mathbf{Z}$ for any $\lambda \geqq 0$.*

PROOF. Suppose that $E(f) = \lambda \mathbf{Z}$ where $\lambda \geqq 0$. Since, by hypothesis, f has bounded variation, applying the Denjoy–Koksma Inequality gives us a $c > 0$ such that $|f_{q_n}(x)| < c$ for all n, and all $x \in \mathbf{T}$. Let ν be the greatest integer such that $\nu\lambda < c$. We choose

$$0 < \varepsilon < \frac{H}{4(2\nu + 1)}$$

where H is the number obtained in 2.7(ii). Consider the compact set

$$K = (-\varepsilon, c] \setminus \bigcup_{i=0}^{\nu} (i\lambda - \varepsilon, i\lambda + \varepsilon).$$

This is the interval $(-\varepsilon, c]$ without intervals of width 2ε about each $i\lambda \in E(f)$, for $i = 0, \dots, \nu$. Clearly $K \cap E(f) = \emptyset$, so applying [S, Proposition 3.8] we obtain a Borel set $B \in \Omega$ with $\mu(B) > 0$ such that $B \cap T^{-m}B \cap \{x : |f_m(x)| \in K\} = \emptyset$ for all $m \in \mathbf{Z}$.

In 2.7(i) we showed that, for all $n > N_1$, every interval Q_i^n in $P_n(\alpha)$ contains a subinterval J_i^n on which $|f_{q_n}|$ rises continuously through a height greater than $\frac{1}{2}H$. We claim that for n sufficiently large, and for every interval Q_i^n, the ratio of the Lebesgue measure of Q_i^n to that of the set $Q_i^n \cap \{x : |f_{q_n}(x)| \in K\}$ is greater than a fixed positive number.

Applying 2.7(ii) to the interval(s) within K, of total length at least $\frac{1}{2}H - (2\nu + 1)\varepsilon > \frac{1}{4}H$, through which the graph of $|f_{q_n}|$ must pass, gives us a strictly positive constant F, independent of i, such that, for $n > N_1$,

$$\frac{\mu(J_i^n \cap \{x : |f_{q_n}(x)| \in K\})}{\mu(Q_i^n)} > F\,\frac{H}{4}.$$

So, by monotonicity, for $n > N_1$ there exists a strictly positive constant W, independent of i, such that

$$\frac{\mu(Q_i^n \cap \{x : |f_{q_n}(x)| \in K\})}{\mu(Q_i^n)} > W. \tag{2.5}$$

This proves our claim.

Now from 2.9, given $0 < \varepsilon' < \frac{1}{2}\mu(B)$, there is a $B_0 \subseteq B$ with $\mu(B \setminus B_0) < \varepsilon'$, and a $\delta > 0$, such that, for all $x \in B_0$,

$$\frac{\mu(I'(x) \cap B)}{\mu(I'(x))} > 1 - \frac{1}{2}W \tag{2.6}$$

for any interval $I'(x)$ containing x, of length $\mu(I'(x)) < \delta$. Then, from 2.8, given $0 < \varepsilon' < \frac{1}{2}\mu(B)$, there is a $B_1 \subseteq B_0$ with $\mu(B_0 \setminus B_1) < \varepsilon'$, and a subsequence $\{q_{k_j}\}$ of partial quotient denominators, with $T^{q_{k_j}}x \in B_0$ for all $x \in B_1$. We now fix $x \in B_1$. From 2.1 we know that, for any i and any $\delta > 0$, there is an M such that, for $k > M$,

$$\mu(Q_i^k(x)) < \delta,$$

where $Q_i^k(x)$ is the interval in $P_n(\alpha)$ which contains x. So for $n = k_j > M$ we have that

$$\frac{\mu(Q_i^n(x) \cap B)}{\mu(Q_i^n(x))} > 1 - \frac{1}{2}W. \tag{2.7}$$

Now, applying (2.6) to $T^{q_n}x \in B_0$ for this n, we also get that

$$\frac{\mu(T^{q_n}Q_i^n(x) \cap B)}{\mu(T^{q_n}Q_i^n(x))} > 1 - \frac{1}{2}W, \tag{2.8}$$

since the interval $T^{q_n}Q_i^n(x)$ encloses the point $T^{q_n}x$ and

$$\mu(T^{q_n}Q_i^n(x)) = \mu(Q_i^n(x)) < \delta.$$

So, using the T-invariance of μ, (2.8) becomes

$$\frac{\mu(Q_i^n(x) \cap T^{-q_n}B)}{\mu(Q_i^n(x))} > 1 - \frac{1}{2}W. \tag{2.9}$$

By (2.7) and (2.9) B and $T^{-q_n}B$ take up, in proportion, more than $1 - \frac{1}{2}W$ of $Q_i^n(x)$ for $n = k_j > M$. Hence

$$\frac{\mu(Q_i^n(x) \cap B \cap T^{-q_n}B)}{\mu(Q_i^n(x))} > 1 - W$$

for $n = k_j > M$.

So, reducing to a subset of B if necessary, we find that, for our chosen n, the set $B \cap T^{-q_n}B$ takes up at least a fixed proportion of $Q_i^n(x)$; from (2.5) we have that, for n sufficiently large, $\{x : |f_{q_n}(x)| \in K\}$ also takes up at least a fixed proportion of $Q_i^n(x)$ for all i. Since these proportions add up to more than 1, the two sets must intersect for $n = k_j > Y = \max(M, N_1)$.

This shows that there is an m with $B \cap T^{-m}B \cap \{x : |f_m(x)| \in K\} \neq \varnothing$. Thus we have a contradiction, and the lemma is proved. □

2.11. THEOREM. *Let f be piecewise absolutely continuous, with $\int f d\mu = 0$, f' Riemann integrable and $\int f' d\mu \neq 0$. Then the skew-product T_f is ergodic.*

PROOF. Since the only closed additive subgroups of $\mathbf{R}$ are $\lambda \mathbf{Z}$ for $\lambda \geqq 0$, or $\mathbf{R}$ itself, then the above lemma and [S, Lemma 3.3] demonstrate that the only possible remaining choice for the essential values is $E(f) = \mathbf{R}$. By [S, Corollary 5.4] this shows that the skew-product T_f is ergodic. □

ACKNOWLEDGEMENTS

The author would like to thank his supervisor, Klaus Schmidt, and Peter Walters for many helpful suggestions and comments.

REFERENCES

[C] P. Cohn, *Measure Theory*, Birkhauser, Boston, 1980.

[HL1] P. Hellekalek and G. Larcher, *On the ergodicity of a class of skew products*, Isr. J. Math. **54** (1986), 301–306.

[HL2] P. Hellekalek and G. Larcher, *On Weyl sums and skew products over irrational rotations*, preprint, Salzburg, 1987.

[K] Y. Katznelson, *Sigma-finite invariant measures for smooth mappings of the circle*, J. Analyse **33** (1977), 1–18.

[O] I. Oren, *Ergodicity of cylinder flows arising from irregularities of distribution*, Isr. J. Math. **44** (1983), 127–138.

[R] W. Rudin, *Fourier Analysis on Groups*, Interscience, Wiley, 1967.

[S] K. Schmidt, *Cocycles of ergodic transformation groups*, Lecture Notes in Mathematics, Vol. 1, MacMillan Co. of India, New Dehli, 1977.